Friday 13 January 2012 – Morning

A2 GCE MATHEMATICS (MEI)

4756 Further Methods for Advanced Mathematics (FP2)

QUESTION PAPER

Candidates answer on the Printed Answer Book.

OCR supplied materials:
- Printed Answer Book 4756
- MEI Examination Formulae and Tables (MF2)

Other materials required:
- Scientific or graphical calculator

Duration: 1 hour 30 minutes

4733190112

INSTRUCTIONS TO CANDIDATES

These instructions are the same on the Printed Answer Book and the Question Paper.

- The Question Paper will be found in the centre of the Printed Answer Book.
- Write your name, centre number and candidate number in the spaces provided on the Printed Answer Book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the Printed Answer Book**. Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions in Section A and **one** question from Section B.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

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- The number of marks is given in brackets **[ ]** at the end of each question or part question on the Question Paper.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- The Printed Answer Book consists of **16** pages. The Question Paper consists of **4** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER / INVIGILATOR

- Do not send this Question Paper for marking; it should be retained in the centre or recycled. Please contact OCR Copyright should you wish to re-use this document.

Section A (54 marks)

Answer all the questions

1 **(a)** A curve has polar equation $r = 1 + \cos\theta$ for $0 \leqslant \theta < 2\pi$.

(i) Sketch the curve. **[2]**

(ii) Find the area of the region enclosed by the curve, giving your answer in exact form. **[6]**

(b) Assuming that x^4 and higher powers may be neglected, write down the Maclaurin series approximations for $\sin x$ and $\cos x$ (where x is in radians).

Hence or otherwise obtain an approximation for $\tan x$ in the form $ax + bx^3$. **[6]**

(c) Find $\displaystyle\int_0^1 \frac{1}{\sqrt{1 - \frac{1}{4}x^2}}\,\mathrm{d}x$, giving your answer in exact form. **[4]**

2 **(a)** The infinite series C and S are defined as follows.

$$C = 1 + a\cos\theta + a^2\cos 2\theta + \ldots,$$

$$S = a\sin\theta + a^2\sin 2\theta + a^3\sin 3\theta + \ldots,$$

where a is a real number and $|a| < 1$.

By considering $C + \mathrm{j}\,S$, show that $C = \dfrac{1 - a\cos\theta}{1 + a^2 - 2a\cos\theta}$ and find a corresponding expression for S. **[8]**

(b) Express the complex number $z = -1 + \mathrm{j}\sqrt{3}$ in the form $r\,\mathrm{e}^{\mathrm{j}\theta}$.

Find the 4th roots of z in the form $r\,\mathrm{e}^{\mathrm{j}\theta}$.

Show z and its 4th roots in an Argand diagram.

Find the product of the 4th roots and mark this as a point on your Argand diagram. **[10]**

3 **(i)** Show that the characteristic equation of the matrix

$$\mathbf{M} = \begin{pmatrix} 3 & -1 & 2 \\ -4 & 3 & 2 \\ 2 & 1 & -1 \end{pmatrix}$$

is $\lambda^3 - 5\lambda^2 - 7\lambda + 35 = 0$. **[4]**

(ii) Show that $\lambda = 5$ is an eigenvalue of $\mathbf{M}$, and find its other eigenvalues. **[4]**

(iii) Find an eigenvector, $\mathbf{v}$, of unit length corresponding to $\lambda = 5$.

State the magnitudes and directions of the vectors $\mathbf{M}^2\mathbf{v}$ and $\mathbf{M}^{-1}\mathbf{v}$. **[6]**

(iv) Use the Cayley-Hamilton theorem to find the constants a, b, c such that

$$\mathbf{M}^4 = a\mathbf{M}^2 + b\mathbf{M} + c\mathbf{I}.$$ **[4]**

Section B (18 marks)

Answer one question

Option 1: Hyperbolic functions

4 **(i)** Define $\tanh t$ in terms of exponential functions. Sketch the graph of $\tanh t$. **[3]**

(ii) Show that $\text{artanh}\, x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$. State the set of values of x for which this equation is valid. **[5]**

(iii) Differentiate the equation $\tanh y = x$ with respect to x and hence show that the derivative of $\text{artanh}\, x$ is $\frac{1}{1-x^2}$.

Show that this result may also be obtained by differentiating the equation in part **(ii)**. **[5]**

(iv) By considering $\text{artanh}\, x$ as $1 \times \text{artanh}\, x$ and using integration by parts, show that

$$\int_0^{\frac{1}{2}} \text{artanh}\, x \, \mathrm{d}x = \frac{1}{4}\ln\frac{27}{16}.$$ **[5]**

Option 2: Investigation of curves

This question requires the use of a graphical calculator.

5 The points A(–1, 0), B(1, 0) and P(x, y) are such that the product of the distances PA and PB is 1. You are given that the cartesian equation of the locus of P is

$$\left((x+1)^2 + y^2\right)\left((x-1)^2 + y^2\right) = 1.$$

(i) Show that this equation may be written in polar form as

$$r^4 + 2r^2 = 4r^2\cos^2\theta.$$

Show that the polar equation simplifies to

$$r^2 = 2\cos 2\theta.$$ **[4]**

(ii) Give a sketch of the curve, stating the values of θ for which the curve is defined. **[4]**

(iii) The equation in part **(i)** is now to be generalised to

$$r^2 = 2\cos 2\theta + k,$$

where k is a constant.

(*A*) Give sketches of the curve in the cases $k = 1$, $k = 2$. Describe how these two curves differ at the pole.

(*B*) Give a sketch of the curve in the case $k = 4$. What happens to the shape of the curve as k tends to infinity? **[7]**

(iv) Sketch the curve for the case $k = -1$.

What happens to the curve as $k \to -2$? **[3]**

THERE ARE NO QUESTIONS WRITTEN ON THIS PAGE.

OCR
RECOGNISING ACHIEVEMENT

Friday 13 January 2012 – Morning

A2 GCE MATHEMATICS (MEI)

4756 Further Methods for Advanced Mathematics (FP2)

PRINTED ANSWER BOOK

Candidates answer on this Printed Answer Book.

OCR supplied materials:
- Question Paper 4756 (inserted)
- MEI Examination Formulae and Tables (MF2)

Other materials required:
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Duration: 1 hour 30 minutes

Candidate forename		Candidate surname	

Centre number						Candidate number				

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Section A (54 marks)

1(a)(i)	
1(a)(ii)	

1(b)

1(c)	

2(a)	
	(answer space continued overleaf)

2(a)	**(continued)**
2(b)	

2(b)	(continued)

3(i)	
3(ii)	

3(iii)	

3(iv)	

Section B (18 marks)

4(i)	
4(ii)	
	(answer space continued overleaf)

4(ii)	**(continued)**
4(iii)	

4(iv)	

5(i)	
5(ii)	

5 (iii) (*A*)	
5 (iii) (*B*)	

5(iv)	

RECOGNISING ACHIEVEMENT

GCE

Mathematics (MEI)

Advanced GCE

Unit **4756:** Further Methods for Advanced Mathematics

Mark Scheme for January 2012

Any enquiries about publications should be addressed to:

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Annotations

Annotation	Meaning
✓and ✗	
BOD	Benefit of doubt
FT	Follow through
ISW	Ignore subsequent working
M0, M1	Method mark awarded 0, 1
A0, A1	Accuracy mark awarded 0, 1
B0, B1	Independent mark awarded 0, 1
SC	Special case
^	Omission sign
MR	Misread
Highlighting	
Other abbreviations in mark scheme	**Meaning**
E1	Mark for explaining
U1	Mark for correct units
G1	Mark for a correct feature on a graph
M1 dep*	Method mark dependent on a previous mark, indicated by *
cao	Correct answer only
oe	Or equivalent
rot	Rounded or truncated
soi	Seen or implied
www	Without wrong working

Subject-specific Marking Instructions

a Annotations should be used whenever appropriate during your marking.

The A, M and B annotations must be used on your standardisation scripts for responses that are not awarded either 0 or full marks. It is vital that you annotate standardisation scripts fully to show how the marks have been awarded.

For subsequent marking you must make it clear how you have arrived at the mark you have awarded.

b An element of professional judgement is required in the marking of any written paper. Remember that the mark scheme is designed to assist in marking incorrect solutions. Correct *solutions* leading to correct answers are awarded full marks but work must not be judged on the answer alone, and answers that are given in the question, especially, must be validly obtained; key steps in the working must always be looked at and anything unfamiliar must be investigated thoroughly.

Correct but unfamiliar or unexpected methods are often signalled by a correct result following an *apparently* incorrect method. Such work must be carefully assessed. When a candidate adopts a method which does not correspond to the mark scheme, award marks according to the spirit of the basic scheme; if you are in any doubt whatsoever (especially if several marks or candidates are involved) you should contact your Team Leader.

c The following types of marks are available.

M
A suitable method has been selected and *applied* in a manner which shows that the method is essentially understood. Method marks are not usually lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, eg by substituting the relevant quantities into the formula. In some cases the nature of the errors allowed for the award of an M mark may be specified.

A
Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated Method mark is earned (or implied). Therefore M0 A1 cannot ever be awarded.

B
Mark for a correct result or statement independent of Method marks.

E
A given result is to be established or a result has to be explained. This usually requires more working or explanation than the establishment of an unknown result.

Unless otherwise indicated, marks once gained cannot subsequently be lost, eg wrong working following a correct form of answer is ignored. Sometimes this is reinforced in the mark scheme by the abbreviation isw. However, this would not apply to a case where a candidate passes through the correct answer as part of a wrong argument.

d When a part of a question has two or more 'method' steps, the M marks are in principle independent unless the scheme specifically says otherwise; and similarly where there are several B marks allocated. (The notation 'dep *' is used to indicate that a particular mark is dependent on an earlier, asterisked, mark in the scheme.) Of course, in practice it may happen that when a candidate has once gone wrong in a part of a question, the work from there on is worthless so that no more marks can sensibly be given. On the other hand, when two or more steps are successfully run together by the candidate, the earlier marks are implied and full credit must be given.

e The abbreviation ft implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A and B marks are given for correct work only — differences in notation are of course permitted. A (accuracy) marks are not given for answers obtained from incorrect working. When A or B marks are awarded for work at an intermediate stage of a solution, there may be various alternatives that are equally acceptable. In such cases, exactly what is acceptable will be detailed in the mark scheme rationale. If this is not the case please consult your Team Leader.

Sometimes the answer to one part of a question is used in a later part of the same question. In this case, A marks will often be 'follow through'. In such cases you must ensure that you refer back to the answer of the previous part question even if this is not shown within the image zone. You may find it easier to mark follow through questions candidate-by-candidate rather than question-by-question.

f Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise. Candidates are expected to give numerical answers to an appropriate degree of accuracy, with 3 significant figures often being the norm. Small variations in the degree of accuracy to which an answer is given (e.g. 2 or 4 significant figures where 3 is expected) should not normally be penalised, while answers which are grossly over- or under-specified should normally result in the loss of a mark. The situation regarding any particular cases where the accuracy of the answer may be a marking issue should be detailed in the mark scheme rationale. If in doubt, contact your Team Leader.

g Rules for replaced work

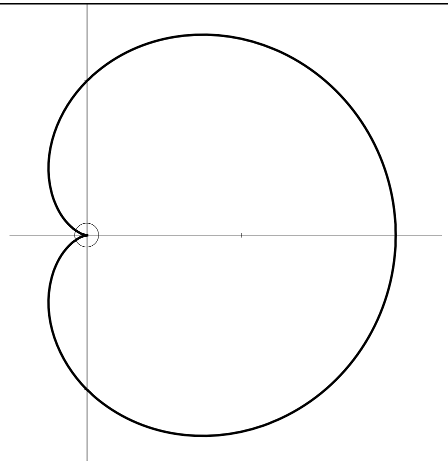
If a candidate attempts a question more than once, and indicates which attempt he/she wishes to be marked, then examiners should do as the candidate requests.

If there are two or more attempts at a question which have not been crossed out, examiners should mark what appears to be the last (complete) attempt and ignore the others.

NB Follow these maths-specific instructions rather than those in the assessor handbook.

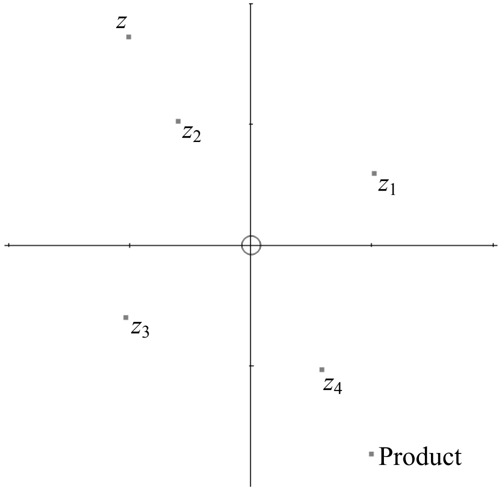
h For a *genuine* misreading (of numbers or symbols) which is such that the object and the difficulty of the question remain unaltered, mark according to the scheme but following through from the candidate's data. A penalty is then applied; 1 mark is generally appropriate, though this may differ for some units. This is achieved by withholding one A mark in the question.

Note that a miscopy of the candidate's own working is not a misread but an accuracy error.

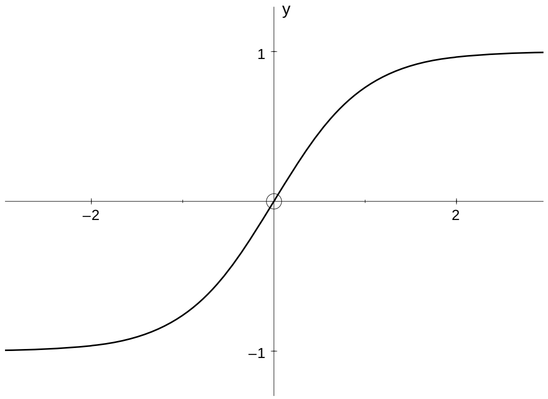
Question			Answer	Marks	Guidance	
1	(a)	(i)		G2 [2]	A fully correct curve Give G1 for one error, e.g. incorrect form at O, lack of clear symmetry, sharp point at RH extremity	
1	(a)	(ii)	Area $=\frac{1}{2}\int_0^{2\pi}(1+\cos\theta)^2\,d\theta$	M1	Integral expression involving $(1 + \cos\theta)^2$	
			$=\frac{1}{2}\int_0^{2\pi}\left(1+2\cos\theta+\cos^2\theta\right)d\theta$	A1	Correct expanded integral expression, incl. limits	Limits may be implied by later work. Penalise missing ½ here (max. 4/6)
			$=\frac{1}{2}\int_0^{2\pi}\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta$	M1	Using $\cos^2\theta=\frac{1}{2}+\frac{1}{2}\cos 2\theta$	Allow sign or factor errors
			$=\frac{1}{2}\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{2\pi}$	A2	Correct result of integration	Give A1 for one error in this expression
			$=\frac{3}{2}\pi$	A1 [6]	Dependent on previous A2	
1	(b)		$\sin x = x-\frac{1}{6}x^3\ldots$ $\cos x = 1-\frac{1}{2}x^2\ldots$	B1	Both series correct as far as second term	Ignore higher-order terms. Allow denominators left as 2!, 3!
			$\tan x \approx \left(x-\frac{1}{6}x^3\right)\left(1-\frac{1}{2}x^2\right)^{-1}$	M1	Using $\tan x=\frac{\sin x}{\cos x}$	Allow even if no further progress but must be used, not just stated

Question			Answer	Marks	Guidance	
			$=\left(x-\frac{1}{6}x^3\right)\left(1+\frac{1}{2}x^2+...\right)$	M1	Using binomial expansion. Dependent on first M1	If methods mixed, mark to benefit of candidate
			$= x+\frac{1}{2}x^3-\frac{1}{6}x^3+...$	M1	Expanding brackets. Dependent on previous M1	
			$= x+\frac{1}{3}x^3...$	A1A1	$a = 1$, $b = \frac{1}{3}$ correctly obtained	Dependent on both M1s. Deduct 1 for each additional term (*)
			OR $\dfrac{x-\frac{1}{6}x^3}{1-\frac{1}{2}x^2}=ax+bx^3$ M1		Using $\tan x=\dfrac{\sin x}{\cos x}$	
			$\Rightarrow x-\frac{1}{6}x^3=\left(1-\frac{1}{2}x^2\right)\left(ax+bx^3\right)=ax+\left(b-\frac{1}{2}a\right)x^3+...$ M1		Attempting to compare coeffs.	
			$\Rightarrow a=1$ A1		Correctly obtained	As (*)
			$b-\frac{1}{2}a=-\frac{1}{6}$ M1		Obtaining b	
			$\Rightarrow b=\frac{1}{3}$ A1		Correctly obtained	As (*)
			OR $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$			
			$f''(x) = 2\sec^2 x\tan x$ M1		Attempting first two derivatives	
			$f'''(x) = 4\sec^2 x\tan^2 x + 2\sec^4 x$ M1		Attempting third derivative	Using the product rule
			$f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = 2$			
			$f(x)=f(0)+xf'(0)+\dfrac{x^2f''(0)}{2!}+...$ M1		Applying Maclaurin series. Dependent on first M1	
			$=x+\frac{1}{3}x^3...$ A1A1		Correctly obtained	As (*)
				[6]		
1	(c)		$\displaystyle\int_0^1\frac{1}{\sqrt{1-\frac{1}{4}x^2}}dx=\int_0^1\frac{2}{\sqrt{4-x^2}}dx=\left[2\arcsin\frac{x}{2}\right]_0^1$	M1	arcsin alone, or any sine substitution	
				A1	2 and $\dfrac{x}{2}$	
			$=2\left(\dfrac{\pi}{6}-0\right)$	M1	Using limits. Dependent on first M1	No need to see explicit use of $x = 0$ Limits wrong way round M0
			$=\dfrac{\pi}{3}$	A1	Evaluated in terms of π	
				[4]		

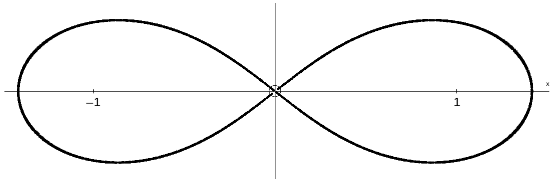
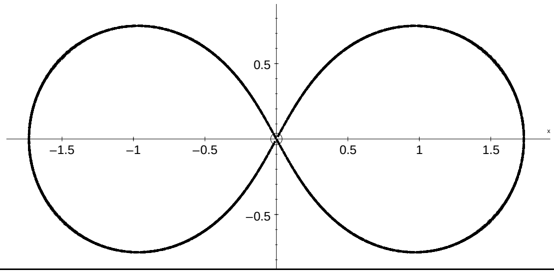
Question			Answer	Marks	Guidance	
2	(a)		$C + jS = 1 + ae^{j\theta} + a^2e^{2j\theta} + \dots$	M1	Forming $C + jS$ as a series of powers	$\dots a^2(\cos 2\theta + j\sin 2\theta)$ insufficient. Powers must be correct
			This is a geometric series with $r = ae^{j\theta}$	M1	Identifying G.P. and attempting sum. Dependent on first M1	
			Sum to infinity $= \frac{1}{1 - ae^{j\theta}}$	A1		
			$= \frac{1}{1-ae^{j\theta}} \times \frac{1-ae^{-j\theta}}{1-ae^{-j\theta}}$	M1*	Multiplying numerator and denominator by $1 - ae^{-j\theta}$ o.e.	
			$= \frac{1-ae^{-j\theta}}{1-ae^{j\theta}-ae^{-j\theta}+a^2}$	M1	Multiplying out denominator. Dependent on M1*	Use of FOIL with powers combined correctly (allow one slip)
			$= \frac{1-a(\cos\theta - j\sin\theta)}{1-2a\cos\theta+a^2}$	M1	Introducing trig functions. Dependent on M1*	Condone e.g. $e^{-j\theta} = \cos\theta + j\sin\theta$
			$= \frac{1-a\cos\theta}{1-2a\cos\theta+a^2} + \frac{aj\sin\theta}{1-2a\cos\theta+a^2}$			
			$\Rightarrow C = \frac{1-a\cos\theta}{1-2a\cos\theta+a^2}$	E1		Answer given. www which leads to C
			and $S = \frac{a\sin\theta}{1-2a\cos\theta+a^2}$	A1		
				[8]		
2	(b)		Modulus = 2	B1		
			Argument $= \frac{2\pi}{3}$	B1		
			$\Rightarrow -1 + j\sqrt{3} = 2e^{j\frac{2\pi}{3}}$			
			$\Rightarrow$ fourth roots have $r = \sqrt[4]{2}$	B1		Allow 1.19 or better
			and $\theta = \frac{\pi}{6}$			
				M1	$\div$ arg z by 4 and adding $\frac{\pi}{2}$	$\theta = \frac{\pi}{6} + \frac{2k\pi}{4}$ scores M1;
			$\Rightarrow$ roots are $\sqrt[4]{2}e^{j\frac{\pi}{6}}, \sqrt[4]{2}e^{j\frac{2\pi}{3}}, \sqrt[4]{2}e^{j\frac{7\pi}{6}}, \sqrt[4]{2}e^{j\frac{5\pi}{3}}$	A1	All arguments correct	$k = 0, 1, 2, 3$ (or $-2, -1, 0, 1$) A1

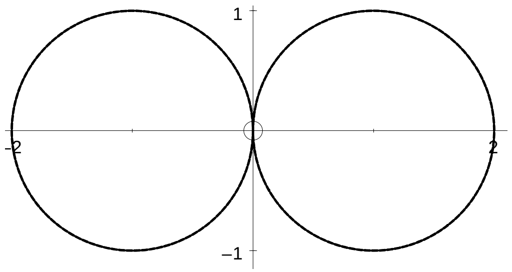
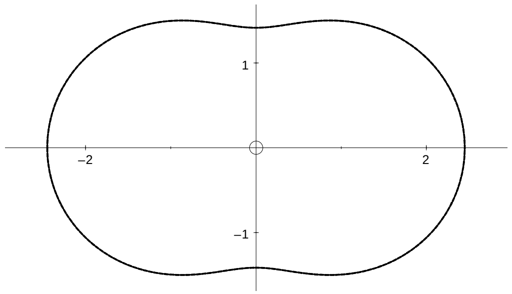
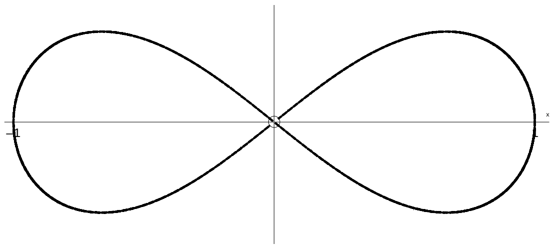
Question			Answer	Marks	Guidance	
				G1 G1ft G1ft	Position of z Roots forming square Position of product	In 2nd quadrant Ignore marked angles Correct or their $-z$
			Product of 4th roots $= 2e^{j(1+4+7+10)\frac{\pi}{6}}$	M1	Attempting to find product	Must consider both modulus and argument
			$= 2e^{j\frac{5\pi}{3}}$	A1 [10]	Or $-\dfrac{\pi}{3}$ o.e.	
3	(i)		$\mathbf{M} - \lambda\mathbf{I} = \begin{pmatrix} 3-\lambda & -1 & 2 \\ -4 & 3-\lambda & 2 \\ 2 & 1 & -1-\lambda \end{pmatrix}$			
			$\det(\mathbf{M} - \lambda\mathbf{I}) = (3-\lambda)[(3-\lambda)(-1-\lambda) - 2]$	M1	Obtaining $\det(\mathbf{M} - \lambda\mathbf{I})$	
			$+ 1[-4(-1-\lambda) - 4] + 2[-4 - 2(3-\lambda)]$	A1	Any correct form	
			$= (3-\lambda)(\lambda^2 - 2\lambda - 5) + 4\lambda + 2(2\lambda - 10)$ $= -\lambda^3 + 5\lambda^2 - \lambda - 15 + 4\lambda + 4\lambda - 20$	M1	Multiplying out. Dep. on first M1	
			$\Rightarrow \lambda^3 - 5\lambda^2 - 7\lambda + 35 = 0$	E1 [4]		Answer given
3	(ii)		$\lambda^3 - 5\lambda^2 - 7\lambda + 35 = 0$	M1	Factorising, obtaining a quadratic	If M0, give B1 for substituting $\lambda = 5$
			$\Rightarrow (\lambda - 5)(\lambda^2 - 7) = 0$	A1	Correct quadratic	
				M1	Solving quadratic	
			$\lambda = \pm\sqrt{7}$	A1 [4]		Allow 2.65 or better

Question			Answer	Marks	Guidance	
3	(iii)		$\lambda = 5 \Rightarrow \begin{pmatrix} -2 & -1 & 2 \\ -4 & -2 & 2 \\ 2 & 1 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$			
			$\Rightarrow -2x - y + 2z = 0$ $-4x - 2y + 2z = 0$ $2x + y - 6z = 0$	M1	Two independent equations	Need to multiply out, unless implied by later work
			$\Rightarrow z = 0, y = -2x$	M1	Obtaining a non-zero eigenvector	
			$\Rightarrow$ eigenvector is $\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$	A1		
			$\Rightarrow$ eigenvector of unit length is $\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$	A1ft	$\frac{1}{\sqrt{5}}$ f.t. their eigenvector	
			$\mathbf{M}^2\mathbf{v}$ has magnitude 25 in direction of $\mathbf{v}$			
			$\mathbf{M}^{-1}\mathbf{v}$ has magnitude $\frac{1}{5}$ in direction of $\mathbf{v}$	B1 B1 [6]	Both magnitudes c.a.o. Directions c.a.o.	 May be given as column vectors
3	(iv)		$\lambda^3 - 5\lambda^2 - 7\lambda + 35 = 0$			
			$\Rightarrow \mathbf{M}^3 - 5\mathbf{M}^2 - 7\mathbf{M} + 35\mathbf{I} = \mathbf{0}$	M1	Using Cayley-Hamilton Theorem	Condone omitted **I**
			$\Rightarrow \mathbf{M}^4 = 5\mathbf{M}^3 + 7\mathbf{M}^2 - 35\mathbf{M}$	A1	Correct expression involving $\mathbf{M}^4$ and non-negative powers of $\mathbf{M}$	
			$= 5(5\mathbf{M}^2 + 7\mathbf{M} - 35\mathbf{I}) + 7\mathbf{M}^2 - 35\mathbf{M}$	M1	Substituting for $\mathbf{M}^3$ and obtaining expression in required form	
			$= 32\mathbf{M}^2 - 175\mathbf{I}$	A1 [4]	$a = 32, b = 0, c = -175$	

Question			Answer	Marks	Guidance	
4	(i)		$\tanh t = \dfrac{e^t - e^{-t}}{e^t + e^{-t}}$	B1	Or $\dfrac{e^{2t}-1}{e^{2t}+1}$	Condone other variables used
				G1	Correct shape	
				G1	Asymptotes at $y = \pm 1$. Dependent on first G1	If text and graph conflict, mark what is shown on the graph
				[3]		
4	(ii)		$y = \text{artanh}\, x \Rightarrow x = \tanh y$			
			$\Rightarrow x = \dfrac{e^y - e^{-y}}{e^y + e^{-y}}$			Or $x = \dfrac{e^{2y}-1}{e^{2y}+1}$
			$\Rightarrow x\left(e^y + e^{-y}\right) = e^y - e^{-y}$	M1	First step in rearrangement	Variables the right way round at some stage, and clearing fractions
			$\Rightarrow xe^y + xe^{-y} = e^y - e^{-y}$			
			$\Rightarrow xe^{-y} + e^{-y} = e^y - xe^y$			
			$\Rightarrow e^{-y}(1+x) = e^y(1-x)$			
			$\Rightarrow e^{2y} = \dfrac{1+x}{1-x}$	M1 A1	Obtaining e^{2y} in terms of x	Dependent on first M1
			$\Rightarrow 2y = \ln\left(\dfrac{1+x}{1-x}\right)$			
			$\Rightarrow \text{artanh}\, x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$	E1		Answer given
			Valid for $-1 < x < 1$	B1 [5]	Independent	

Question			Answer	Marks	Guidance	
4	(iii)		$\tanh y = x$ $\Rightarrow \operatorname{sech}^2 y \frac{dy}{dx} = 1$ $\Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y} = \frac{1}{1-\tanh^2 y}$ $= \frac{1}{1-x^2}$ $y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x)$ $\Rightarrow \frac{dy}{dx} = \frac{1}{2}\times\frac{1}{1+x} - \frac{1}{2}\times\frac{-1}{1-x}$ $= \frac{1}{2}\times\frac{1-x+1+x}{(1+x)(1-x)}$ $= \frac{1}{1-x^2}$	M1 E1 M1 A1 E1 [5]	Differentiating and explicitly attempting to express in terms of tanh y Correctly obtained Attempting logarithmic diff. Any correct form Convincing manipulation	
4	(iv)		$\int_0^{\frac{1}{2}} \operatorname{artanh} x\, dx = [x \operatorname{artanh} x]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{1-x^2}\, dx$ $= \frac{1}{2}\operatorname{artanh}\frac{1}{2} - \left[-\frac{1}{2}\ln(1-x^2)\right]_0^{\frac{1}{2}}$ $= \frac{1}{4}\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) + \frac{1}{2}\ln\frac{3}{4}$ $= \frac{1}{4}\ln 3 + \frac{1}{4}\ln\frac{9}{16}$ $= \frac{1}{4}\ln\frac{27}{16}$	M1 A1 A1 M1 E1 [5]	Using integration by parts This line correct $-\frac{1}{2}\ln(1-x^2)$ Applying limits and using (ii) in result of integration Convincing manipulation www	With $u = \operatorname{artanh} x$, $v' = 1$ and $v = x$ Condone omitted limits Or $-\frac{1}{2}\ln(1-x) - \frac{1}{2}\ln(1+x)$ Must be exact Answer given

Question			Answer	Marks	Guidance	
5	(i)		$((x+1)^2+y^2)((x-1)^2+y^2)=1$ $\Rightarrow (x^2+2x+1+y^2)(x^2-2x+1+y^2)=1$ $\Rightarrow (r^2+2r\cos\theta+1)(r^2-2r\cos\theta+1)=1$ $\Rightarrow (r^2+1)^2-4r^2\cos^2\theta=1$ $\Rightarrow r^4+2r^2=4r^2\cos^2\theta$ $\Rightarrow r^4=2r^2(2\cos^2\theta-1)$ $\Rightarrow r^4=2r^2\cos 2\theta$ $\Rightarrow r^2=2\cos 2\theta$	M1 E1 M1 E1 [4]	Using $r^2=x^2+y^2$ and $x=r\cos\theta$ Correctly obtained Using $\cos 2\theta = 2\cos^2\theta - 1$ Correctly obtained	
5	(ii)		$r^2 \ge 0 \Rightarrow \cos 2\theta \ge 0$ $\Rightarrow 0 \le \theta \le \frac{\pi}{4},\ \frac{3\pi}{4} \le \theta \le \frac{5\pi}{4},\ \frac{7\pi}{4} \le \theta \le 2\pi$ 	M1 A1 G2 [4]	Considering $\cos 2\theta \ge 0$ Or B2. Inequalities must be non-strict Curve must be complete. Award G1 for a curve with one error	Or $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4},\ \frac{3\pi}{4} \le \theta \le \frac{5\pi}{4}$
5	(iii)	(*A*)	$k=1$: 	G1	Curve must be complete	

Question			Answer	Marks	Guidance	
			$k = 2$:  For $k = 1$, the gradients at the pole are finite For $k = 2$, they appear to be infinite	G2 B1	Curve must be complete. Award G1 for a curve with one error	For G2, curve must appear sufficiently different from case $k = 1$
		(*B*)	$k = 4$:  Tends to circle as k tends to infinity	G2 B1 [7]	Curve must be complete. Award G1 for a curve with one error	
5	(iv)		$k = -1$:  As $k \to -2$, the curve retains its figure-of-eight shape, but contracts towards the origin	G2 B1 [3]	Curve must be complete. Award G1 for a curve with one error	

4756 Further Methods for Advanced Mathematics (FP2)

General Comments

The overall standard of work was most impressive, with over 30% of candidates scoring 60 marks or more, and fewer than 5% scoring 20 marks or fewer. Question 1 (calculus) and Question 3 (matrices) were the best done questions, with Question 2 (complex numbers) close behind: indeed, there was a marked improvement in work on complex numbers this series. Question 4 (hyperbolic functions) was found the most difficult by some margin, while very few candidates attempted the alternative Question 5 (investigations of curves).

For future series candidates would be well advised to look for simpler methods: the quadratic formula is not the most appropriate way to solve $\lambda^2 - 7 = 0$. Also, enough detail should be given in working to convince the examiner that the candidate has validly obtained a given answer: full marks (or, indeed, very many marks at all) will not be given if it appears "as if by magic".

This was the first series in which the paper was marked on-line, and a printed answer book used. This caused no problems. A small number of candidates used the blank spaces for Question 5 to work on other questions; it is better if they use additional sheets.

Comments on Individual Questions

1 (Calculus: polar curves, Maclaurin series, standard integral)
The mean mark on this question was about 14 out of 18.

1 (a) The cardioid in (i) was usually correct, although some could have been bigger! Errors included a sharp point at the right-hand extremity and missing the pole altogether.

In (ii), the area of the curve was very well done, and there were many concise and efficient solutions. Most knew how to deal with the integral of $\cos^2\theta$ although a few gave $\cos^3\theta/3$, and the integral of $\cos\theta$ was given as $-\sin\theta$ fairly frequently.

1 (b) Although many correct answers were seen, this was the least well done part of question 1. Many candidates wasted time by deriving the Maclaurin series for sin x and cos x, although the instruction in the question was "write down". Then most realised that they had to divide sin x by cos x, although a few attempted to divide cos x by sin x. However, having obtained $\dfrac{x - \frac{1}{6}x^3}{1 - \frac{1}{2}x^2}$, many then began to differentiate tan x repeatedly: this was often managed correctly and the required result obtained. Comparatively few reached the approximation for tan x by writing the quotient as $\left(x - \frac{1}{6}x^3\right)\left(1 - \frac{1}{2}x^2\right)^{-1}$ and using the binomial expansion.

1 (c) All but a very few realised that this was an arcsin integral: the most common error was to obtain $\frac{1}{2}\arcsin\frac{x}{2}$ rather than $2\arcsin\frac{x}{2}$ as the result. A few candidates obtained $\arcsin 2x$ and then appeared surprised that they could not evaluate $\arcsin 2$.

2 (Complex numbers: infinite series, fourth roots)
The mean mark on this question was about 13 out of 18. Work on this topic appeared significantly better than in previous series.

2 (a) Most candidates could recognise *C* + *jS* as a geometric series, and sum it to infinity, although a substantial number produced a formula for the sum to *n* terms. It was particularly pleasing to see many candidates explicitly checking that the sum to infinity existed.

Many stopped at this point, but there was a pleasing improvement in the number of candidates who were able to realise the denominator. Those who left their expressions in terms of exponential functions as late as possible generally had less trouble with the manipulation than those who introduced trigonometric functions earlier. A very common error was to give the 1 from the numerator as part of *S*, which is the imaginary part, and/or get the sign wrong.

2 (b) The response to this part-question on fourth roots of a complex number was most encouraging, with well over half the candidates scoring 9 or 10 out of 10. Efficient methods were used. Common errors included: omitting *z* from the Argand diagram; giving the argument of *z* as ±π/3 or 5π/6; and quoting the modulus of the product of the fourth roots as $4 \times \sqrt[4]{2}$. A few found the sum of the roots instead.

3 (Matrices: characteristic equation, eigenvalues, eigenvectors and the Cayley-Hamilton theorem)
The mean mark on this question was about 14 out of 18.

3 (i) Well over 80% of candidates scored full marks in this part. A variety of methods were seen, including Sarrus' method and even the elegant use of elementary operations to produce a zero in an appropriate place before finding the determinant.

3 (ii) Virtually all candidates obtained the quadratic factor $\lambda^2 - 7$. Solving $\lambda^2 - 7 = 0$ caused more of a problem! The quadratic formula was often used and a few candidates gave only the positive root, or quoted the roots as ±7.

3 (iii) This part caused the most problems. Most knew the method to obtain an eigenvector, although having obtained *y* = −2*x* many gave the eigenvector as $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ or similar, as has happened in previous series. Relatively few candidates took account of the instruction "find an eigenvector...of unit length". The intention in the last section was that candidates would observe that $\mathbf{M}^2\mathbf{v} = 25\mathbf{v}$ etc. but relatively few candidates did this: indeed, many calculated $\mathbf{M}^2$ and $\mathbf{M}^{-1}$ and worked from there, even performing trigonometrical feats to produce directions in terms of angles which were inappropriate in three dimensions anyway.

3 (iv) The Cayley-Hamilton theorem was well known and this part was generally very well done, although arithmetic and sign errors were quite common. Just a few calculated $\mathbf{M}^2$ as a 3 × 3 matrix and tried to work from there.

4 (Hyperbolic functions: tanh and artanh)
This question caused the most difficulty: the mean mark was about 10 out of 18. Fewer than 10% of candidates scored full marks. Many candidates dropped many or all of their *h*s, writing tanh as tan etc. This was condoned!

4 (i) The vast majority knew the correct exponential form of tanh t and could draw the graph, although not all included the important information that it is bounded by $y = \pm 1$.

4 (ii) Many carried this out very efficiently but some new "laws" of logarithms were invented. A substantial number used the quadratic formula to find e^y from $e^{2y}(1-x)-(1+x)=0$, which led to manipulation and sign issues. Some forgot to give the range of validity but for other candidates this gained them their only mark in this part. Quite a few candidates started with $\text{artanh}\, x = \frac{\text{arsinh}\, x}{\text{arcosh}\, x}$ and their logarithmic equivalents, which was often followed by copious quantities of manipulation leading magically to the "correct" answer.

4 (iii) This part was not well done. Many carried out the instruction to differentiate tanh $y = x$ but could not relate $\text{sech}^2 y$ to $1 - x^2$. Some got as far as $\cosh^2(\text{artanh}\, x)$ which they then tried to express in exponential form. This usually succeeded only in filling the answer space, although some completely correct solutions by this route were seen.

The differentiation of the logarithmic form given in part (ii) was rarely carried through correctly or efficiently. The easiest way is to use a (correct) log law to split the logarithm but many tried to differentiate $\frac{1+x}{1-x}$ using the quotient rule and got into a tangle, making various manipulation and sign errors.

4 (iv) The vast majority used integration by parts appropriately but integrating $\frac{x}{1-x^2}$ caused considerable difficulty: indeed, many stopped at this point. Those who went further often did use the result in part (ii) to introduce logarithms, but often needed to show much more detail when producing the given answer.

5 (Investigations of curves)

Few candidates attempted this question, but some good answers were seen, especially to part (i).

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